

# Recursive Encoding and Decoding of Noiseless Subsystem and Decoherence Free Subspace

Chi-Kwong Li

*Department of Mathematics, College of William & Mary, Williamsburg,  
VA 23187-8795, USA. (Year 2011: Department of Mathematics,  
Hong Kong University of Science & Technology, Hong Kong.)\**

Mikio Nakahara

*Research Center for Quantum Computing, Interdisciplinary Graduate School of Science and Engineering,  
and Department of Physics, Kinki University, Kinki University,  
3-4-1 Kowakae, Higashi-Osaka, 577-8502, Japan.†*

Yiu-Tung Poon

*Department of Mathematics, Iowa State University, Ames, IA 50051, USA.‡*

Nung-Sing Sze

*Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Hong Kong.§*

Hiroyuki Tomita

*Research Center for Quantum Computing, Interdisciplinary Graduate School of Science and Engineering,  
Kinki University, 3-4-1 Kowakae, Higashi-Osaka, 577-8502, Japan.¶*

When the environmental disturbance to a quantum system has a wavelength much larger than the system size, all qubits localized within a small area are under action of the same error operators. Noiseless subsystem and decoherence free subspace are known to correct such collective errors. We construct simple quantum circuits, which implement these collective error correction codes, for a small number  $n$  of physical qubits. A single logical qubit is encoded with  $n = 3$  and  $n = 4$ , while two logical qubits are encoded with  $n = 5$ . The recursive relations among the subspaces employed in noiseless subsystem and decoherence free subspace play essential rôles in our implementation. The recursive relations also show that the number of gates required to encode  $m$  logical qubits increases linearly in  $m$ .

## I. INTRODUCTION

A quantum system is vulnerable to external noise. In quantum information processing and quantum computation, the system must be protected from the environmental noise one way or another to protect information stored in the quantum registers. The majority of quantum systems employed for these purposes is microscopic in size, typically on the order of a few microns. In contrast, the environmental noise, such as electromagnetic wave, has the wavelength on the order of a few centimeters or more. Therefore, it is natural to assume all the qubits in the register suffer from the same error operator. We call such error the collective error in the following. Suppose  $n$ -qubit quantum states  $\rho$  are represented as  $N \times N$  density matrices with  $N = 2^n$ , and a quantum channel is realized as a completely positive

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\*Electronic address: ckli@math.wm.edu

†Electronic address: nakahara@math.kindai.ac.jp

‡Electronic address: ytpoon@iastate.edu

§Electronic address: raymond.sze@inet.polyu.edu.hk

¶Electronic address: tomita@alice.math.kindai.ac.jp

linear map  $\Phi$  with an operator sum representation

$$\Phi(\rho) = \sum_{j=1}^r E_j \rho E_j^* \quad (1)$$

for the error operators  $E_1, \dots, E_r$ ; see [1, 2]. Then the error operators of our channel can be expressed as multiples of operator of the form  $W^{\otimes n} \in \mathbf{2}^{\otimes n}$ , where  $\mathbf{2}$  is the two-dimensional (fundamental) irreducible representation of  $SU(2)$ .

Decoherence free subspace [3–6] and noiseless subsystem [7–10] are two standard methods to correct collective errors; see [10, 11]. It is not hard to explain the scheme using the operator sum representation of the quantum channel (1) as follows. Suppose the finite dimensional  $C^*$ -algebra  $\mathcal{A}$  generated by the error operators admits the unique decomposition into irreducible representations up to unitary equivalence (similarity) as

$$\bigoplus_j (I_{r_j} \otimes M_{n_j}) \quad \text{with} \quad \sum_j r_j n_j = N,$$

where  $n_j$  is the dimension of the irreducible representation while  $r_j$  its multiplicity. Then every error operator  $E_i$  in (1) has the form

$$\bigoplus_j (I_{r_j} \otimes B_j) \quad \text{with} \quad B_j \in M_{n_j}.$$

For every index  $j$ , if we regard

$$M_N = (I_{r_j} \otimes M_{n_j}) \oplus M_q, \quad q = N - r_j n_j,$$

and if apply the channel to a quantum state  $\rho = (\hat{\rho} \otimes \sigma) \oplus O_q$  with  $\hat{\rho} \in M_{r_j}$  and  $\sigma \in M_{n_j}$ , according to this decomposition, then

$$\Phi(\rho) = (\hat{\rho} \otimes \sigma_E) \oplus O_q$$

because of the special form of the error operators in this decomposition. Here  $O_q$  is a null matrix of order  $q$ . Thus, the state  $\hat{\rho} \in M_{r_j}$  encoded as above will not be affected by the errors (noise) and can be easily recovered. This gives rise to a noiseless subsystem. The situation is particularly pleasant if  $n_j = 1$ , i.e., we use the one dimensional irreducible representation of  $\mathcal{A}_n$ , so that

$$\Phi(\hat{\rho} \oplus O_q) = \hat{\rho} \oplus O_q.$$

In such a case, we get a decoherence free subspace.

We are interested in an efficient construction, which leads to simple implementation, of decoherence free subspaces and noiseless subsystems for the channels with common error on each qubit in the register. By the discussion in the preceding paragraph, construction of decoherence free subspace employs one-dimensional irreducible representations of the algebra  $\mathcal{A}_n$  generated by  $\mathbf{2}^{\otimes n}$  for encoding while the latter encodes logical qubits by making use of the multiplicity of some irreducible representations.

It is the purpose of this paper to investigate the implementation of these ideas in terms of the quantum circuits. We consider decoherence free subspace with  $n = 4$ , which implements a single logical qubit and noiseless subsystem with  $n = 3$  and 5, which encodes a single logical qubit and two logical qubits, respectively. Viola *et al* [12] worked out the circuit implementation of  $n = 3$  noiseless subsystem and demonstrated its validity by using ion trap quantum computer. No further works have been conducted for  $n \geq 4$  to date to our knowledge. Our implementation, starting

with  $n = 3$  noiseless subsystem, is recursive so that  $n = 4$  decoherence subsystem and  $n = 5$  noiseless subsystem are implemented with the quantum circuit for  $n = 3$ . Moreover, our circuit for  $n = 3$  is simpler than that obtained by Yang and Gea-Banacloche [9] and by Viola *et al* [12].

We construct a quantum circuit for  $n = 3$  noiseless subsystem in the next section. We analyze  $n = 4$  decoherence free subspace and  $n = 5$  noiseless subsystem in Sections III and IV by making use of the result of Section II. Our analysis is concrete and encoding basis vectors and quantum circuits are explicitly constructed. The last section is devoted to summary and discussion.

We will use the known fact (see [10]) that the algebra  $\mathcal{A}_n$  generated by  $\mathbf{2}^{\otimes n}$  has the unique decomposition

$$\bigoplus_{0 \leq j \leq n/2} (I_{r_j} \otimes M_{n_j})$$

with  $(r_0, n_0) = (1, n + 1)$  and  $(r_j, n_j) = \left( \binom{n}{j} - \binom{n}{j-1}, n + 1 - 2j \right)$  for  $0 < j \leq n/2$ . Also, we will employ the Lie theoretic notation and regard a qubit belonging to the fundamental representation  $\mathbf{2}$  of  $SU(2)$  while the product operator  $W^{\otimes n}$  acts as a reducible representation of  $SU(2)^{\otimes n}$ , denoted by  $\mathbf{2}^{\otimes n}$ .

## II. 3-QUBIT NOISELESS SUBSYSTEM

Let us consider a 3-qubit system and see how it can be used to encode a logical qubit which is robust against any noise of the form  $W^{\otimes 3}$ , where  $W$  is an arbitrary element of the fundamental representation  $\mathbf{2}$ . To this end, we first consider the algebra  $\mathcal{A}_3$  of  $\mathbf{2}^{\otimes 3}$ .  $\mathcal{A}_3$  is decomposed into the sum of irreducible representations as

$$\mathbf{2}^{\otimes 3} = \mathbf{4} \oplus (I_2 \otimes \mathbf{2}),$$

where  $I_n$  is the unit matrix of dimension  $n$ . Corresponding to this decomposition, any unitary matrix  $V \in \mathbf{2}^{\otimes 3}$  can be decomposed as

$$V = V_4 \oplus (I_2 \otimes V_2)$$

under a proper choice of basis vectors. Here  $V_4$  belongs to  $\mathbf{4}$  and  $V_2$  to  $\mathbf{2}$  of  $SU(2)$ . It should be noted that  $I_2$  is immune to any collective noise of the form  $W^{\otimes 3}$ ,  $W \in \mathbf{2}$  and the corresponding vector space form the noiseless subsystem.

The success of our schemes depends on a judicious choice of orthonormal basis for the decomposition of the algebra  $\mathcal{A}$  generated by  $\mathbf{2}^{\otimes 3}$ . To this end, let  $\{|e_{4,1}\rangle, |e_{4,2}\rangle, |e_{4,3}\rangle, |e_{4,4}\rangle\}$  be a basis of  $\mathbf{4}$ ,  $\{|e_{a1}\rangle, |e_{a2}\rangle\}$  and  $\{|e_{b1}\rangle, |e_{b2}\rangle\}$  be the bases of the two  $\mathbf{2}$  defined as follows [13].

$$\begin{cases} |e_{4,1}\rangle = |000\rangle, \\ |e_{4,2}\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle), \\ |e_{4,3}\rangle = \frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle), \\ |e_{4,4}\rangle = |111\rangle, \end{cases} \quad (2)$$

$$\begin{cases} |e_{a1}\rangle = \frac{1}{\sqrt{2}}(|100\rangle - |010\rangle), \\ |e_{a2}\rangle = -\frac{1}{\sqrt{2}}(|011\rangle - |101\rangle), \end{cases} \quad (3)$$

$$\begin{cases} |e_{b1}\rangle = \frac{1}{\sqrt{6}}(|100\rangle + |010\rangle - 2|001\rangle), \\ |e_{b2}\rangle = -\frac{1}{\sqrt{6}}(|011\rangle + |101\rangle - 2|110\rangle). \end{cases} \quad (4)$$

We implement a noiseless subsystem from two  $\mathbf{2}$  representations.

Suppose  $U_E^{(3)}$  is an encoding matrix which generates the above basis vectors from the binary basis vectors  $|i_1 i_2 i_3\rangle$ , ( $i_k \in \{0, 1\}$ ). We choose  $U_E^{(3)}$  to have columns

$$(|e_{a1}\rangle, |e_{b1}\rangle, |e_{a2}\rangle, |e_{b2}\rangle, |e_{a3}\rangle, |e_{b3}\rangle, |e_{a4}\rangle, |e_{b4}\rangle)$$

in this order.

**Theorem II.1** *Let  $\alpha, \beta, \gamma$  be any real numbers and let*

$$X_\alpha = (e^{i\alpha\sigma_x})^{\otimes 3}, Y_\beta = (e^{i\beta\sigma_y})^{\otimes 3}, Z_\gamma = (e^{i\gamma\sigma_z})^{\otimes 3},$$

where  $\sigma_k$ 's are the Pauli matrices. Consider a quantum channel  $\Phi : M_8 \rightarrow M_8$  given by

$$\Phi(\rho) = p_0 \rho + p_1 X_\alpha \rho X_\alpha^\dagger + p_2 Y_\beta \rho Y_\beta^\dagger + p_3 Z_\gamma \rho Z_\gamma^\dagger$$

for some  $p_i \in \mathbb{R}$  such that  $\sum_{i=0}^3 p_i \leq 1$ . Then for any data state  $\hat{\rho} \in M_2$ ,  $U_E^{(3)}$  and  $\Phi$  satisfy the identity

$$U_E^{(3)\dagger} \Phi \left( U_E^{(3)} (\rho_a \otimes |0\rangle\langle 0| \otimes \hat{\rho}) U_E^{(3)\dagger} \right) U_E^{(3)} = \left( \sum_{j=0}^3 p_j U_j \rho_a U_j^\dagger \right) \otimes |0\rangle\langle 0| \otimes \hat{\rho}, \quad (5)$$

that is, the initial data state is recovered in the output state with no entanglement with the ancilla qubits. Here  $\rho_a$  is an initial single qubit ancilla state and

$$U_0 = I_2, \quad U_1 = e^{i\alpha\sigma_x}, \quad U_2 = e^{i\beta\sigma_y}, \quad U_3 = e^{i\gamma\sigma_z}.$$

Proof: We show that the  $\mathbf{2} \oplus \mathbf{2}$  irreducible representations form a noiseless subsystem by explicit evaluation. Let  $\{|e_{a1}\rangle, |e_{a2}\rangle\}$  spans the logical  $|0\rangle_L$  state, while  $\{|e_{b1}\rangle, |e_{b2}\rangle\}$  spans the logical  $|1\rangle_L$  state. We show that noise operators  $X_\alpha, Y_\beta$  and  $Z_\gamma$  leave each subspace invariant.

Let  $P_a = \sum_{i=1}^2 |e_{ai}\rangle\langle e_{ai}|$  and  $P_b = \sum_{i=1}^2 |e_{bi}\rangle\langle e_{bi}|$ . Then it is easy to show

$$X_\alpha P_k X_\alpha^\dagger = Y_\beta P_k Y_\beta^\dagger = Z_\gamma P_k Z_\gamma^\dagger = P_k \quad (k = a, b).$$

It should be noted that, although the whole four-dimensional subsystem is invariant under  $X_\alpha, Y_\beta$  and  $Z_\gamma$ , we cannot use this subsystem to encode two-qubit state since each vector is not invariant under the action of the error operators.

Now it is easy to prove the identity. We use a pure state notation to simplify the expressions. The general case with mixed initial states  $\rho_a$  and  $\hat{\rho}$  is obtained by simply mixing the pure state results using linearity. Let  $|\hat{\psi}\rangle = a|0\rangle + b|1\rangle$  be a data qubit state to be encoded and  $|v\rangle = v_0|0\rangle + v_1|1\rangle$  be the initial state of the first ancilla qubit, while that of the second qubit is set to  $|0\rangle$ . Under the action of  $U_E^{(3)}$ , along with a two qubit state  $|v\rangle|0\rangle$ ,  $|\hat{\psi}\rangle$  is encoded as

$$|\Psi\rangle = U_E^{(3)} |v\rangle|0\rangle|\hat{\psi}\rangle = v_0(a|e_{a1}\rangle + b|e_{b1}\rangle) + v_1(a|e_{a2}\rangle + b|e_{b2}\rangle).$$

Let us consider a noise operator  $X_\alpha$  first. Its action on  $|\Psi\rangle$  yields

$$\begin{aligned} |\Psi_X\rangle &= X_\alpha |\Psi\rangle \\ &= (v_0 \cos \alpha + i v_1 \sin \alpha)(a|e_{a1}\rangle + b|e_{b1}\rangle) + (v_1 \cos \alpha + i v_0 \sin \alpha)(a|e_{a2}\rangle + b|e_{b2}\rangle). \end{aligned}$$

The action of the recovery operator  $U_E^{(3)\dagger}$  recovers the initial state, except for the first qubit, as

$$U_E^{(3)\dagger}|\Psi_X\rangle = (e^{i\alpha\sigma_x}|v\rangle)|0\rangle|\hat{\psi}\rangle,$$

which shows that data qubit state is immune to  $X_\alpha$ . It is shown similarly that the data qubit is immune to other error operators either. Since each error is in action with the probability  $p_i$ , we have proved the identity (5). ■

A remark is in order. In contrast with an ordinary QECC, the scheme corrects multiple action of the error operators. It was shown in the theorem that the top-most qubit can be any superposition state or mixed state initially and its output state is another superposition/mixed state under an action of a single collective error operator in  $X_\alpha, Y_\beta$  and  $Z_\gamma$ . It should be noted that the error channel leaves the encoded word unchanged. Namely, given any initial ancilla state  $\rho_a$ , there exists an ancilla state  $\rho'_a$  such that

$$\Phi\left(U_E^{(3)}(\rho_a \otimes |0\rangle\langle 0| \otimes \hat{\rho})U_E^{(3)\dagger}\right) = U_E^{(3)}(\rho'_a \otimes |0\rangle\langle 0| \otimes \hat{\rho})U_E^{(3)\dagger}.$$

Then the error correction may be repeated as many times as required. This implies that it corrects any error operator of the form  $W^{\otimes 3}$ , where  $W \in \mathbf{2}$ . This is because any element  $W \in \mathbf{2}$  of  $\text{SU}(2)$  is decomposed into a product

$$W = e^{i\theta_1\sigma_x}e^{i\theta_2\sigma_y}e^{i\theta_3\sigma_x}.$$

It should be clear that  $W^{\otimes 3}$  is expressed as a product  $X_{\theta_1}Y_{\theta_2}X_{\theta_3}$ , each factor of which leaves the noiseless subsystem invariant.

One of the simplest quantum circuits which implement the encoding matrix  $U_E^{(3)}$  is obtained by simple redefinitions of the basis vectors;

$$\begin{aligned} |e_{a1}\rangle &= \frac{1}{\sqrt{2}}(|100\rangle - |001\rangle), \\ |e_{b1}\rangle &= \frac{1}{\sqrt{6}}(|100\rangle + |001\rangle - 2|010\rangle), \\ |e_{4,2}\rangle &= |111\rangle, \\ |e_{4,1}\rangle &= \frac{1}{\sqrt{3}}(|100\rangle + |001\rangle + |010\rangle), \\ |e_{a2}\rangle &= -(\sigma_x)^{\otimes 3}|e_{a1}\rangle, \quad |e_{b2}\rangle = -(\sigma_x)^{\otimes 3}|e_{b1}\rangle, \\ |e_{4,3}\rangle &= -(\sigma_x)^{\otimes 3}|e_{4,2}\rangle, \quad |e_{4,4}\rangle = -(\sigma_x)^{\otimes 3}|e_{4,1}\rangle. \end{aligned}$$

A permutation of the basis vectors takes much simpler form with the redefined basis and the quantum circuit is found by inspection. Figure 1 shows an example of the encoding circuit, in which  $G_1$  and  $G_2$  stand for

$$G_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}, \quad G_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Note that our circuit is simpler than that found in [9] and [12] regarding the number of gates.

### III. 4-QUBIT DECOHERENCE FREE SUBSPACE

We design the 4-qubit decoherence free subspace, which is robust against collective noise of the form  $W^{\otimes 4}$  ( $W \in \mathbf{2}$ ), by taking advantage of the noiseless subsystem analyzed in the previous section.

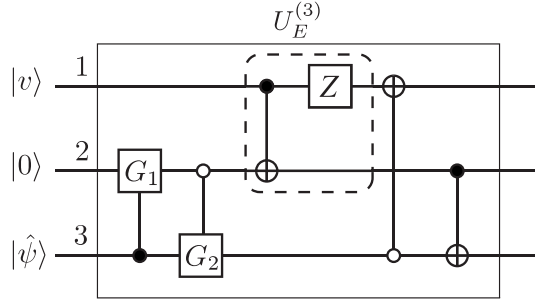


FIG. 1: Encoding circuit  $U_E^{(3)}$  of the noiseless subsystem for a 3-qubit system. The filled (empty) circle attached to the control qubit denotes that the gate acts on the target qubit when the control qubit is set to  $|1\rangle$  ( $|0\rangle$ ), respectively, and otherwise it is left alone. It encodes a single qubit state  $|\hat{\psi}\rangle$ . See text for redefinition of basis vectors to simplify the circuit. The part surrounded by a broken line can be omitted if the initial state of the top-most qubit is  $|0\rangle$ , which makes the circuit even simpler. The recovery operation is given by  $U_E^{(3)\dagger}$ . This circuit is employed as a module in the implementation of the noiseless subsystem and the decoherence free subspace for larger  $n$ .

A 4-qubit system is used to encode a logical qubit which is robust against any collective noise. The algebra  $\mathcal{A}_4$  obtained from  $\mathbf{2}^{\otimes 4}$  is decomposed into the sum of irreducible representations;

$$\mathbf{2}^{\otimes 4} = \mathbf{5} \oplus (I_3 \otimes \mathbf{3}) \oplus (I_2 \otimes \mathbf{1}).$$

Corresponding to this decomposition, any unitary matrix  $V \in \mathbf{2}^{\otimes 4}$  can be decomposed as

$$V = V_5 \oplus (I_3 \otimes V_3) \oplus (I_2 \otimes V_1)$$

under a proper choice of basis vectors. Here  $V_k$  belongs to the irreducible representation  $\mathbf{k}$ ,  $k = 1, 3, 5$  of  $SU(2)$ . It should be noted that the singlet irreducible representation is immune to any operator  $V = W^{\otimes 4}$ ,  $W \in \mathbf{2}$  and two of them form a single logical qubit which is immune to any noise of the form  $V$ . This vector space robust against collective noise is called the decoherence free subspace (DFS).

We generate basis vectors  $|S = 0, S_z = 0\rangle$  of two one-dimensional representations of  $SU(2)$  from  $\{|e_{ai}\rangle, |e_{bi}\rangle\}$  as

$$\begin{aligned} |0\rangle_L &= \frac{1}{\sqrt{2}}(|1\rangle|e_{a1}\rangle - |0\rangle\Sigma_x|e_{a1}\rangle) = \frac{1}{\sqrt{2}}(|1\rangle|e_{a1}\rangle + |0\rangle(\sigma_x)^{\otimes 3}|e_{a1}\rangle), \\ |1\rangle_L &= \frac{1}{\sqrt{2}}(|1\rangle|e_{b1}\rangle - |0\rangle\Sigma_x|e_{b1}\rangle) = \frac{1}{\sqrt{2}}(|1\rangle|e_{b1}\rangle + |0\rangle(\sigma_x)^{\otimes 3}|e_{b1}\rangle), \end{aligned}$$

where  $\Sigma_x = \sum_{i=1}^3 \sigma_x^i = -\sigma_x^{\otimes 3}$  for  $S = 1/2$ . It is important in the implementation of the encoding circuit to realize that

$$\begin{aligned} |0\rangle_L &= (X \otimes I_8)(\text{CNNN})(H \otimes I_8)|0\rangle \otimes |e_{a1}\rangle = (X \otimes I_8)(\text{CNNN})(H \otimes U_E^{(3)})|0\rangle \otimes |000\rangle, \\ |1\rangle_L &= (X \otimes I_8)(\text{CNNN})(H \otimes I_8)|0\rangle \otimes |e_{b1}\rangle = (X \otimes I_8)(\text{CNNN})(H \otimes U_E^{(3)})|0\rangle \otimes |001\rangle, \end{aligned}$$

where CNNN is a controlled NOT gate with one control bit (the top-most qubit) and three target bits (the rest of the qubits).

Figure 2 shows an example of the encoding circuit for the four-qubit DFS. In contrast with the three-qubit noiseless subsystem, the second qubit (the first input qubit of  $U_E^{(3)}$ ) must be initially set to  $|0\rangle$  for successful encoding of the DFS in the present case.

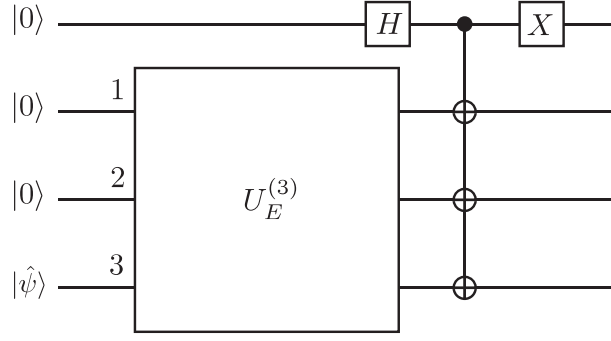


FIG. 2: Encoding circuit  $U_E^{(4)}$  of the decoherence free subspace for a 4-qubit system. It encodes a single qubit state  $|\hat{\psi}\rangle$ . The recovery operation is given by  $U_E^{(4)\dagger}$ .

#### IV. 5-QUBIT NOISELESS SUBSYSTEM

Noiseless subsystem using five qubits encodes two data qubits. It is recursively implemented by employing the encoding circuit  $U_E^{(3)}$  for the three-qubit noiseless subsystem.

The algebra  $\mathcal{A}_5$  obtained from  $2^{\otimes 5}$  is decomposed into the sum of irreducible representations as

$$2^{\otimes 5} = \mathbf{6} \oplus (I_4 \otimes \mathbf{4}) \oplus (I_5 \otimes \mathbf{2}).$$

Corresponding to this decomposition, any unitary matrix  $V \in 2^{\otimes 5}$  is decomposed as

$$V = V_6 \oplus (I_4 \otimes V_4) \oplus (I_5 \otimes V_2)$$

under a proper choice of basis vectors. Here  $V_k$  belongs to the irreducible representation  $\mathbf{k}$ ,  $k = 2, 4, 6$  of  $SU(2)$ . We implement a noiseless subsystem by employing the five two-dimensional representation spaces.

Let  $\{|e_{ai}\rangle, |e_{bi}\rangle\}$  be basis vectors introduced in Section II. We generate four basis vectors  $\{|00\rangle_L, |01\rangle_L, |10\rangle_L, |11\rangle_L\}$  from four two-dimensional representations of  $SU(2)$  as

$$\begin{aligned} |00\rangle_L &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)|e_{a1}\rangle, \\ |01\rangle_L &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)|e_{b1}\rangle, \\ |10\rangle_L &= \frac{1}{\sqrt{6}}(|01\rangle + |10\rangle)|e_{a1}\rangle - 2|00\rangle|e_{a2}\rangle, \\ |11\rangle_L &= \frac{1}{\sqrt{6}}(|01\rangle + |10\rangle)|e_{b1}\rangle - 2|00\rangle|e_{b2}\rangle. \end{aligned}$$

It is important to realize the self-similar structure between the above basis vectors and those of the 3-qubit noiseless subsystem. The third qubit basis vectors in the latter case is replaced by the logical qubit basis vectors of the 3-qubit noiseless subsystem in the above basis vectors. This observation makes implementation of the encoding/decoding circuit almost a trivial work. Note that we do not need to worry about the rest of the basis vectors so far as they are orthogonal to the above basis vectors spanning the noiseless subsystem and that this orthogonalization is automatically taken into account if we employ the unitary matrix  $U_E^{(3)}$  for implementation.

Figure 3 shows an example of the encoding circuit of the five-qubit noiseless subsystem. It should be noted that the top-most qubit can be any state while all the other encoding ancilla qubits must be in the state  $|0\rangle$ . Each  $U_E^{(3)}$

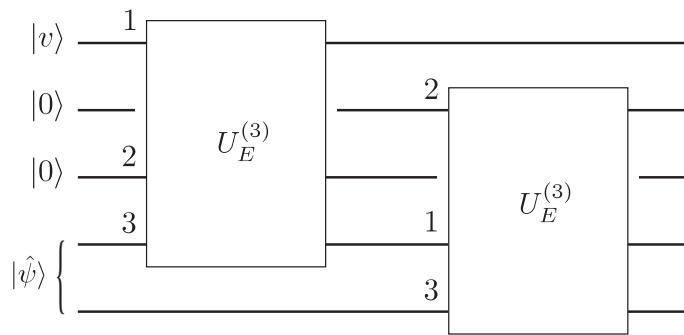


FIG. 3: Encoding circuit  $U_E^{(5)}$  of the 5-qubit noiseless subsystem, which encodes a two data qubit state  $|\hat{\psi}\rangle$ .

acts on the three qubits numbered 1, 2 and 3, which are fed into the input ports 1,2 and 3, respectively, in Fig. 1. The qubit line passing underneath the gate  $U_E^{(3)}$  is not affected by  $U_E^{(3)}$ .

## V. SUMMARY AND DISCUSSIONS

Decoherence free subspace (DFS) and noiseless subsystem make use of vector subspaces which are immune to collective noise of the form  $W^{\otimes n}$ , where  $W$  belongs to **2** of  $SU(2)$ . We have constructed simple encoding and decoding quantum circuits of noiseless subsystem for  $n = 3$  and 5 and DFS for  $n = 4$ . Our strategy is to use the encoding/decoding circuit  $U_E^{(3)}$  for  $n = 3$  recursively in the implementation for  $n = 4$  and  $n = 5$ .

It can be shown generally that  $m$  logical qubits are implemented with  $(2m+1)$ -qubit and  $(2m+2)$ -qubit systems by the same recursive implementations. It should be clear from our construction that  $m$  logical qubits are implemented by use of  $m$   $U_E^{(3)}$ -modules, which shows that the circuit complexity for our encoding and decoding circuits increases merely linearly in  $m$ .

Note, however, that our construction is not the most economical one. There are  $\binom{n}{m} - \binom{n}{m-1}$  basis vectors in 2-dimensional irreducible representations for  $n = 2m+1$ , which encode  $k = \lfloor \log_2 \left( \binom{n}{m} - \binom{n}{m-1} \right) \rfloor$  qubits. This number  $k$  is greater than  $m$  for  $n \geq 9$ , and actually  $k/n \rightarrow 1$  as  $n \rightarrow \infty$ . This asymptotic behavior is also observed in [10] for DFS.

It was shown that the top-most qubit in Figs. 1 and 3 can be any state. Although the entropy of the qubit system increases in general, it remains constant if the top-most qubit is maximally mixed initially as  $\rho_a = \frac{1}{2}I_2$ . This state is attained after operations of many random unitary errors  $W^{\otimes n}$ , for example. This behavior is somewhat analogous to DFS with  $\rho_a = |0\rangle\langle 0|$ , in which the entropy does not change at all.

## Acknowledgement

CKL was supported by a USA NSF grant, a HK RGC grant, the 2011 Fulbright Fellowship, and the 2011 Shanxi 100 Talent Program. He is an honorary professor of University of Hong Kong, Taiyuan University of Technology, and Shanghai University. MN and HT were supported by “Open Research Center” Project for Private Universities: matching fund subsidy from MEXT (Ministry of Education, Culture, Sports, Science and Technology). MN would like to thank partial supports of Grants-in-Aid for Scientific Research from the JSPS (Grant No. 23540470). YTP



was supported by a USA NSF grant. NSS was supported by a HK RGC grant.

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  - [13] Let  $|s, m\rangle$  denote the eigenvector of  $S^2$  and  $S_z$  with eigenvalues  $s(s+1)$  and  $m$ , respectively. Then  $|e_{4,k}\rangle$  corresponds to  $|\frac{3}{2}, \frac{5}{2} - k\rangle$  and  $|e_{a(b)k}\rangle$  to  $|\frac{1}{2}, \frac{3}{2} - k\rangle$ . Explicitly, they are  $|e_{4,1}\rangle = |\frac{3}{2}, \frac{3}{2}\rangle$ ,  $|e_{4,2}\rangle = |\frac{3}{2}, \frac{1}{2}\rangle$ ,  $|e_{4,3}\rangle = |\frac{3}{2}, -\frac{1}{2}\rangle$ ,  $|e_{4,4}\rangle = |\frac{3}{2}, -\frac{3}{2}\rangle$ ,  $|e_{a1}\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$ ,  $|e_{a2}\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$ ,  $|e_{b1}\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$ ,  $|e_{b2}\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$ .